

The finite volume method for diffusion problems

Introduction

Here we develop the numerical method based on, the *finite volume (or control volume) method*, by considering the simplest transport process of all: pure diffusion in the steady state. The governing equation of steady diffusion can easily be derived from the general transport equation for property ϕ by deleting the transient and convective terms.

This gives

$$\text{div}(\Gamma \text{ grad } \phi) + S_\phi = 0$$

The control volume integration, which forms the key step of the finite volume method that distinguishes it from all other CFD techniques, yields the following form:

$$\begin{aligned} \int_{\text{CV}} \text{div}(\Gamma \text{ grad } \phi) dV + \int_{\text{CV}} S_\phi dV \\ = \int_A \mathbf{n} \cdot (\Gamma \text{ grad } \phi) dA + \int_{\text{CV}} S_\phi dV = 0 \end{aligned}$$

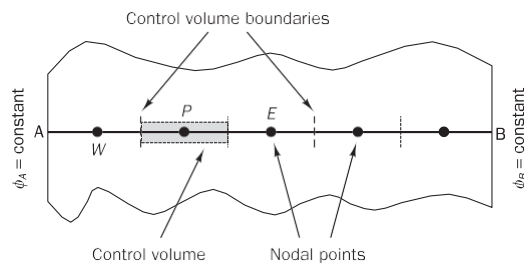
The approximation techniques that are needed to obtain the so-called discretised equations are introduced. Application of the method to simple one-dimensional steady state heat transfer problems is illustrated through a series of worked examples, and the accuracy of the method is gauged by comparing numerical results with analytical solutions.

Finite volume method for one-dimensional steady state diffusion

Consider the steady state diffusion of a property ϕ in a one-dimensional domain defined in the following Figure. The process is governed by

$$\frac{d}{dx} \left(\Gamma \frac{d\phi}{dx} \right) + S = 0$$

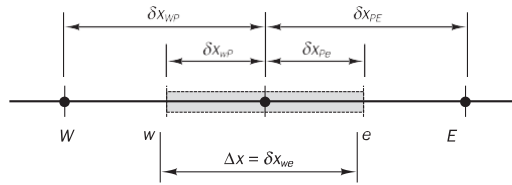
where Γ is the diffusion coefficient and S is the source term. Boundary values of ϕ at points A and B are prescribed. An example of this type of process, one-dimensional heat conduction in a rod, is studied in detail in the following sections.



Step 1: Grid generation

The first step in the finite volume method is to divide the domain into discrete control volumes. Let us place a number of nodal points in the space between A and B. The boundaries (or faces) of control volumes are positioned mid-way between adjacent nodes. Thus each node is surrounded by a control volume or cell. It is common practice to set up control volumes near the edge of the domain in such a way that the physical boundaries coincidewith the control volume boundaries.

At this point it is appropriate to establish a system of notation that can be used in future developments. The usual convention of CFD methods is shown in Figure.



A general nodal point is identified by P and its neighbours in a one-dimensional geometry, the nodes to the west and east, are identified by W and E respectively. The west side face of the control volume is referred to by w and the east side control volume face by e . The distances between the nodes W and P , and between nodes P and E , are identified by δx_{WP} and δx_{PE} respectively. Similarly distances between face w and point P and between P and face e are denoted by δx_{wP} and δx_{Pe} respectively. Figure 4.2 shows that the control volume width is $\Delta x = \delta x_{we}$.

Step 2: Discretisation

The key step of the finite volume method is the integration of the governing equation (or equations) over a control volume to yield a discretised equation at its nodal point P . For the control volume defined above this gives

$$\int_{\Delta V} \frac{d}{dx} \left(\Gamma \frac{d\phi}{dx} \right) dV + \int_{\Delta V} S dV = \left(\Gamma A \frac{d\phi}{dx} \right)_e - \left(\Gamma A \frac{d\phi}{dx} \right)_w + \bar{S} \Delta V = 0$$

Here A is the cross-sectional area of the control volume face, ΔV is the volume and \bar{S} is the average value of source S over the control volume. It is a very attractive feature of the finite volume method that the discretised equation has a clear physical interpretation. The Equation states that the diffusive flux of ϕ leaving the east face minus the diffusive flux of ϕ entering the west face is equal to the generation of ϕ , i.e. it constitutes a balance equation for ϕ over the control volume.

In order to derive useful forms of the discretised equations, the interface diffusion coefficient Γ and the gradient $d\phi/dx$ at east (e) and west (w) are required. Following well-established practice, the values of the property ϕ and the diffusion coefficient are defined and evaluated at nodal points. To calculate gradients (and hence fluxes) at the control volume faces an approximate distribution of properties between nodal points is used. Linear approximations seem to be the obvious and simplest way of calculating interface values and the gradients. This practice is called central differencing. In a uniform grid linearly interpolated values for Γ_w and Γ_e are given by

$$\Gamma_w = \frac{\Gamma_W + \Gamma_P}{2}$$

$$\Gamma_e = \frac{\Gamma_P + \Gamma_E}{2}$$

And the diffusive flux terms are evaluated as

$$\left(\Gamma A \frac{d\phi}{dx} \right)_e = \Gamma_e A_e \left(\frac{\phi_E - \phi_P}{\delta x_{PE}} \right)$$

$$\left(\Gamma A \frac{d\phi}{dx} \right)_w = \Gamma_w A_w \left(\frac{\phi_P - \phi_W}{\delta x_{WP}} \right)$$

In practical situations, as illustrated later, the source term S may be a function of the dependent variable. In such cases the finite volume method approximates the source term by means of a linear form:

$$\check{S}\Delta V = S_u + S_p \phi_P$$

Substitution of equations the two hereabove reported equations into the previous one gives

$$\Gamma_e A_e \left(\frac{\phi_E - \phi_P}{\delta x_{PE}} \right) - \Gamma_w A_w \left(\frac{\phi_P - \phi_W}{\delta x_{WP}} \right) + (S_u + S_p \phi_P) = 0$$

This can be rearranged as

$$\left(\frac{\Gamma_e}{\delta x_{PE}} A_e + \frac{\Gamma_w}{\delta x_{WP}} A_w - S_p \right) \phi_P = \left(\frac{\Gamma_w}{\delta x_{WP}} A_w \right) \phi_W + \left(\frac{\Gamma_e}{\delta x_{PE}} A_e \right) \phi_E + S_u$$

Identifying the coefficients of ϕ_W and ϕ_E in equation as a_W and a_E , and the coefficient of ϕ_P as a_P , the above equation can be written as

$$a_P \phi_P = a_W \phi_W + a_E \phi_E + S_u$$

where

a_W	a_E	a_P
$\frac{\Gamma_w}{\delta x_{WP}} A_w$	$\frac{\Gamma_e}{\delta x_{PE}} A_e$	$a_W + a_E - S_p$

The values of S_u and S_p can be obtained from the source model: $\check{S}\Delta V = S_u + S_p \phi_P$. Equations above represent the discretised form of the first equation seen in the chapter. This type of discretised equation is central to all further developments.

Step 3: Solution of equations

Discretised equations must be set up at each of the nodal points in order to solve a problem. For control volumes that are adjacent to the domain boundaries the general discretised equation is modified to incorporate boundary conditions. The resulting system of linear algebraic equations is then solved to obtain the distribution of the property ϕ at nodal points. Any suitable matrix solution technique may be enlisted for this task.

Worked examples: one-dimensional steady state diffusion

The application of the finite volume method to the solution of simple diffusion problems involving conductive heat transfer is presented in this section. The equation governing one-dimensional steady state conductive heat transfer is

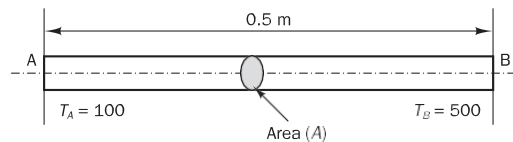
$$\frac{d}{dx} \left(k \frac{dT}{dx} \right) + S = 0$$

where thermal conductivity k takes the place of Γ and the dependent variable is temperature T . The source term can, for example, be heat generation due to an electrical current passing through the rod. Incorporation of boundary conditions as well as the treatment of source terms will be introduced by means of three worked examples.

Consider the problem of source-free heat conduction in an insulated rod whose ends are maintained at constant temperatures of 100°C and 500°C respectively. The one-dimensional problem sketched in Figure is governed by

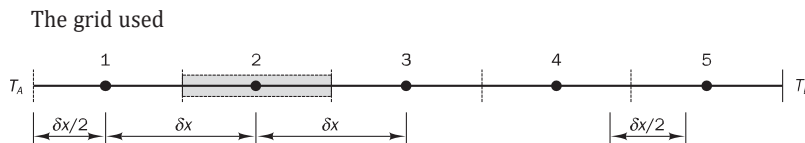
$$\frac{d}{dx} \left(k \frac{dT}{dx} \right) = 0$$

Calculate the steady state temperature distribution in the rod. Thermal conductivity k equals 1000 W/m.K , cross-sectional area A is $10 \times 10^{-3} \text{ m}^2$.



Solution

Let us divide the length of the rod into five equal control volumes as shown in Figure. This gives $\delta x = 0.1 \text{ m}$.



The grid consists of five nodes. For each one of nodes 2, 3 and 4 temperature values to the east and west are available as nodal values. Consequently, discretised equations can be readily written for control volumes surrounding these nodes:

$$\left(\frac{k_e}{\delta x_{PE}} A_e + \frac{k_w}{\delta x_{WP}} A_w \right) T_P = \left(\frac{k_w}{\delta x_{WP}} A_w \right) T_W + \left(\frac{k_e}{\delta x_{PE}} A_e \right) T_E$$

The thermal conductivity ($k_e = k_w = k$), node spacing (δx) and cross-sectional area ($A_e = A_w = A$) are constants. Therefore the **discretised equation for nodal points 2, 3 and 4** is

$$a_P T_P = a_W T_W + a_E T_E$$

with

a_W	a_E	a_P
$\frac{k}{\delta x} A$	$\frac{k}{\delta x} A$	$a_W + a_E$

S_u and S_p are zero in this case since there is no source term in the governing equation.

Nodes 1 and 5 are boundary nodes, and therefore require special attention. Integration of the equation over the control volume surrounding node 1 gives

$$kA \left(\frac{T_E - T_P}{\delta x} \right) - kA \left(\frac{T_P - T_A}{\delta x/2} \right) = 0$$

This expression shows that the flux through control volume boundary A has been approximated by assuming a linear relationship between temperatures at boundary point A and node P. We can rearrange it as follows:

$$\left(\frac{k}{\delta x} A + \frac{2k}{\delta x} A \right) T_P = 0 \cdot T_W + \left(\frac{k}{\delta x} A \right) T_E + \left(\frac{2k}{\delta x} A \right) T_A$$

It can be easily identified that the fixed temperature boundary condition enters the calculation as a source term ($S_u + S_p T_P$) with $S_u = (2kA/\delta x)T_A$ and $S_p = -2kA/\delta x$, and that the link to the (west) boundary side has been suppressed by setting coefficient a_w to zero.

The hereabove equation may be cast as to yield the **discretised equation for boundary node 1**:

$$a_P T_P = a_W T_W + a_E T_E + S_u$$

with

a_W	a_E	a_P	S_p	S_u
0	$\frac{kA}{\delta x}$	$a_W + a_E - S_p$	$-\frac{2kA}{\delta x}$	$\frac{2kA}{\delta x} T_A$

The control volume surrounding node 5 can be treated in a similar manner. Its discretised equation is given by

$$kA \left(\frac{T_B - T_P}{\delta x/2} \right) - kA \left(\frac{T_P - T_W}{\delta x} \right) = 0$$

As before we assume a linear temperature distribution between node P and boundary point B to approximate the heat flux through the control volume boundary. Equation can be rearranged as

$$\left(\frac{k}{\delta x} A + \frac{2k}{\delta x} A \right) T_P = \left(\frac{k}{\delta x} A \right) T_W + 0 \cdot T_E + \left(\frac{2k}{\delta x} A \right) T_B$$

The discretised equation for boundary node 5 is

$$a_P T_P = a_W T_W + a_E T_E + S_u$$

where

a_W	a_E	a_P	S_P	S_u
$\frac{kA}{\delta x}$	0	$a_W + a_E - S_P$	$-\frac{2kA}{\delta x}$	$\frac{2kA}{\delta x} T_B$

The discretisation process has yielded one equation for each of the nodal points 1 to 5. Substitution of numerical values gives $kA/\delta x = 100$, and the coefficients of each discretised equation can easily be worked out. Their values are given in Table 4.1.

The resulting set of algebraic equations for this example is

$$\begin{aligned} 300T_1 &= 100T_2 + 200T_A \\ 200T_2 &= 100T_1 + 100T_3 \\ 200T_3 &= 100T_2 + 100T_4 \\ 200T_4 &= 100T_3 + 100T_5 \\ 300T_5 &= 100T_4 + 200T_B \end{aligned}$$

Table

Node	a_W	a_E	S_u	S_P	$a_P = a_W + a_E - S_P$
1	0	100	$200T_A$	-200	300
2	100	100	0	0	200
3	100	100	0	0	200
4	100	100	0	0	200
5	100	0	$200T_B$	-200	300

This set of equations can be rearranged as

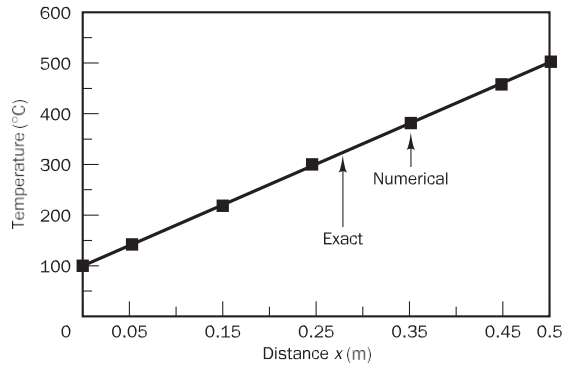
$$\begin{bmatrix} 300 & -100 & 0 & 0 & 0 \\ -100 & 200 & -100 & 0 & 0 \\ 0 & -100 & 200 & -100 & 0 \\ 0 & 0 & -100 & 200 & -100 \\ 0 & 0 & 0 & -100 & 300 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 200T_A \\ 0 \\ 0 \\ 0 \\ 200T_B \end{bmatrix}$$

The above set of equations yields the steady state temperature distribution of the given situation. For simple problems involving a small number of nodes the resulting matrix equation can easily be solved with a software package such as MATLAB (1992). For $T_A = 100$ and $T_B = 500$ the solution of the matrix can be obtained by using, for example, Gaussian elimination:

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 140 \\ 220 \\ 300 \\ 380 \\ 460 \end{bmatrix}$$

The exact solution is a linear distribution between the specified boundary temperatures: $T = 800x + 100$. The subsequent Figure shows that the exact solution and the numerical results coincide.

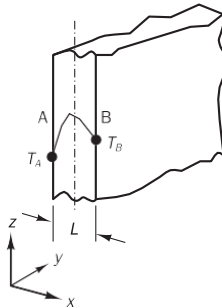
Comparison of the numerical result with the analytical solution



Example 2

Now we discuss a problem that includes sources other than those arising from boundary conditions. Next Figure shows a large plate of thickness $L = 2$ cm with constant thermal conductivity $k = 0.5$ W/m.K and uniform heat generation $q = 1000$ kW/m³. The faces A and B are at temperatures of 100°C and 200°C respectively. Assuming that the dimensions in the y- and z-directions are so large that temperature gradients are significant in the x-direction only, calculate the steady state temperature distribution. Compare the numerical result with the analytical solution. The governing equation is

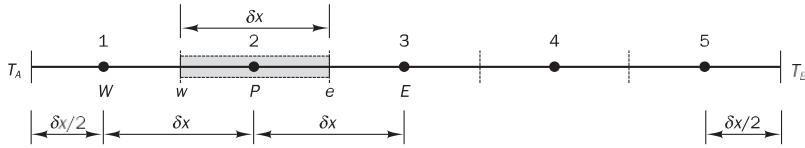
$$\frac{d}{dx} \left(k \frac{dT}{dx} \right) + q = 0$$



Solution

As before, the method of solution is demonstrated using a simple grid. The domain is divided into five control volumes (see next Figure), giving $\delta x = 0.004$ m; a unit area is considered in the y-z plane.

The grid used



Formal integration of the governing equation over a control volume gives

$$\int_{\Delta V} \frac{d}{dx} \left(k \frac{dT}{dx} \right) dV + \int_{\Delta V} q dV = 0$$

We treat the first term of the above equation as in the previous example. The second integral, the source term of the equation, is evaluated by calculating the average generation (i.e. $\bar{q} \Delta V = q \Delta V$) within each control volume. The Equation can be written as

$$\left[\left(k_e A \frac{dT}{dx} \right)_e - \left(k_w A \frac{dT}{dx} \right)_w \right] + q \Delta V = 0$$

$$\left[k_e A \left(\frac{T_E - T_P}{\delta x} \right) - k_w A \left(\frac{T_P - T_W}{\delta x} \right) \right] + q A \delta x = 0$$

The above equation can be rearranged as

$$\left(\frac{k_e A}{\delta x} + \frac{k_w A}{\delta x} \right) T_P = \left(\frac{k_w A}{\delta x} \right) T_W + \left(\frac{k_e A}{\delta x} \right) T_E + q A \delta x$$

This equation is written in the general form of:

$$a_P T_P = a_W T_W + a_E T_E + S_u$$

Since $k_e = k_w = k$ we have the following coefficients:

a_W	a_E	a_P	S_P	S_u
$\frac{kA}{\delta x}$	$\frac{kA}{\delta x}$	$a_W + a_E - S_P$	0	$qA \delta x$

The equation is valid for control volumes at **nodal points 2, 3 and 4**.

To incorporate the boundary conditions at nodes 1 and 5 we apply the linear approximation for temperatures between a boundary point and the adjacent nodal point. At node 1 the temperature at the west boundary is known. Integration of equation at the control volume surrounding node 1 gives

$$\left[\left(kA \frac{dT}{dx} \right)_e - \left(kA \frac{dT}{dx} \right)_w \right] + q\Delta V = 0$$

Introduction of the linear approximation for temperatures between A and P
 Yields

$$\left[k_e A \left(\frac{T_E - T_P}{\delta x} \right) - k_A A \left(\frac{T_P - T_A}{\delta x/2} \right) \right] + qA\delta x = 0$$

The above equation can be rearranged, using $k_e = k_A = k$, to yield the discretised equation for **boundary node 1**:

$$a_P T_P = a_W T_W + a_E T_E + S_u$$

where

a_W	a_E	a_P	S_P	S_u
0	$\frac{kA}{\delta x}$	$a_W + a_E - S_P$	$-\frac{2kA}{\delta x}$	$qA\delta x + \frac{2kA}{\delta x} T_A$

At nodal point 5, the temperature on the east face of the control volume is known. The node is treated in a similar way to boundary node 1. At boundary point 5 we have

$$\left[\left(kA \frac{dT}{dx} \right)_e - \left(kA \frac{dT}{dx} \right)_w \right] + q\Delta V = 0$$

$$\left[k_B A \left(\frac{T_B - T_P}{\delta x/2} \right) - k_w A \left(\frac{T_P - T_W}{\delta x} \right) \right] + qA\delta x = 0$$

The above equation can be rearranged, noting that $k_B = k_w = k$, to give the discretised equation for **boundary node 5**:

$$a_P T_P = a_W T_W + a_E T_E + S_u$$

where

a_W	a_E	a_P	S_P	S_u
$\frac{kA}{\delta x}$	0	$a_W + a_E - S_P$	$-\frac{2kA}{\delta x}$	$qA\delta x + \frac{2kA}{\delta x} T_B$

Substitution of numerical values for $A = 1$, $k = 0.5 \text{ W/m.K}$, $q = 1000 \text{ kW/m}^3$ and $\delta x = 0.004 \text{ m}$ everywhere gives the coefficients of the discretised equations summarised in next Table .

Table

Node	a_W	a_E	S_u	S_P	$a_P = a_W + a_E - S_P$
1	0	125	$4000 + 250T_A$	-250	375
2	125	125	4000	0	250
3	125	125	4000	0	250
4	125	125	4000	0	250
5	125	0	$4000 + 250T_B$	-250	375

Given directly in matrix form the equations are

$$\begin{bmatrix} 375 & -125 & 0 & 0 & 0 \\ -125 & 250 & -125 & 0 & 0 \\ 0 & -125 & 250 & -125 & 0 \\ 0 & 0 & -125 & 250 & -125 \\ 0 & 0 & 0 & -125 & 375 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 29000 \\ 4000 \\ 4000 \\ 4000 \\ 54000 \end{bmatrix}$$

The solution to the above set of equations is

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 150 \\ 218 \\ 254 \\ 258 \\ 230 \end{bmatrix}$$

Comparison with the analytical solution

The analytical solution to this problem may be obtained by integrating the above equation twice with respect to x and by subsequent application of the boundary conditions. This gives

$$T = \left[\frac{T_B - T_A}{L} + \frac{q}{2k}(L - x) \right] x + T_A$$

The comparison between the finite volume solution and the exact solution is shown in the subsequent Table and Figure and it can be seen that, even with a coarse grid of five nodes, the agreement is very good.

Table

Node number	1	2	3	4	5
x (m)	0.002	0.006	0.01	0.014	0.018
Finite volume solution	150	218	254	258	230
Exact solution	146	214	250	254	226
Percentage error	2.73	1.86	1.60	1.57	1.76

Comparison of the numerical result with the analytical solution

