

## 2. KINEMATIC ANALYSIS OF RIGID BODIES

A rigid body that can move in a 3D space has six degrees of freedom (d.o.f.s), that is the three components of the translational displacements,  $x$ - $y$ - $z$ , evaluated along the axes of Cartesian coordinate system and the three rotations,  $\theta_x$ - $\theta_y$ - $\theta_z$ , about the above mentioned axes  $x$ - $y$ - $z$ . Conversely, a rigid body that can be only subjected to planar motions has three degrees of freedom (d.o.f.s), that is, the two components of the translational displacements,  $x$ - $y$ , evaluated along the axes of a 2D Cartesian coordinate system and the rotation  $\theta_z$  about an axis,  $z$ , orthogonal to the plane in which the body can move.

Let us consider a rigid body (Figure 2.1) that is moving in a plane, for the sake of simplicity. The Cartesian  $x$ - $y$  reference system, having origin in  $O$ , is fixed. The straight line  $r$  is fixed to the rigid body. The position of the rigid body can be completely determined by assigning the coordinates,  $x$  and  $y$ , of a generic point  $A$  of the body and the angular position of the straight line  $r$ , with respect to a reference axis (e.g. horizontal).

In a general case study the rigid body can be subjected to a rotor-translating *instantaneous* motion, that, at any time  $t$ , can be always decomposed into an infinitesimal (or finite) rigid translation and an infinitesimal (or finite) pure rotation about an axis orthogonal to the plane  $x$ - $y$ .

Let us consider two points,  $A$  and  $B$  of this rigid body (Figure 2.2). The distance  $|\mathbf{BA}| = |(\mathbf{B}-\mathbf{A})|$  is constant. The absolute position vectors of the points  $A$  and  $B$ , defined with respect to a fixed Cartesian reference system  $x$ - $y$  are given by:

$$(\mathbf{A}-\mathbf{O}) = |(\mathbf{A}-\mathbf{O})| e^{i\alpha} = (i x_A + j y_A) \quad (\mathbf{B}-\mathbf{O}) = |(\mathbf{B}-\mathbf{O})| e^{i\beta} \quad (2.1)$$

The relative position of  $B$ , with respect to  $A$ , is given by the vector:

$$(\mathbf{B}-\mathbf{A}) = (\mathbf{B}-\mathbf{O}) - (\mathbf{A}-\mathbf{O}) = |(\mathbf{B}-\mathbf{O})| e^{i\beta} - |(\mathbf{A}-\mathbf{O})| e^{i\alpha} \quad (\mathbf{B}-\mathbf{A}) = |(\mathbf{B}-\mathbf{A})| e^{i\vartheta} \quad (2.2)$$

Therefore, absolute position of  $B$ , can be expressed as:

$$(\mathbf{B}-\mathbf{O}) = (\mathbf{B}-\mathbf{A}) + (\mathbf{A}-\mathbf{O}) = |(\mathbf{B}-\mathbf{A})| e^{i\vartheta} + |(\mathbf{A}-\mathbf{O})| e^{i\alpha} \quad (2.3)$$

That is:

$$(\mathbf{B}-\mathbf{O}) = |(\mathbf{B}-\mathbf{O})| e^{i\beta} = |(\mathbf{B}-\mathbf{A})| e^{i\vartheta} + (i x_A + j y_A) \quad (2.4)$$

The coordinates  $x_A(t)$  and  $y_A(t)$ , along with the angle  $\vartheta(t)$ , are the three degrees of freedom of the rigid body (the motion of which can only occur in a 2D space). Let us denote  $\mathbf{v}_A$  and  $\mathbf{a}_A$  the vectors of the absolute velocity and acceleration of  $A$ , evaluated at time  $t$ . These two kinematic parameters are assumed to be known, along with the distance  $|\mathbf{BA}|$  and the angle  $\vartheta$ .

The absolute velocity of  $B$  can be obtained by differentiating the eq.(2.4):

$$\mathbf{v}_B = \rho \dot{\vartheta} e^{i(\vartheta+\pi/2)} + \mathbf{v}_A = \rho \omega e^{i(\vartheta+\pi/2)} + \mathbf{v}_A = \mathbf{v}_r + \mathbf{v}_t \quad (2.5)$$

where  $\rho$  is the constant distance  $|\mathbf{BA}|$  and  $\omega$  is the instantaneous angular speed with which the rigid body rotates, at time  $t$  ( $\omega = \dot{\vartheta}$ ), see Figures 2.3 and 2.4.

This result can be also obtained considering a translational Cartesian reference system, the origin of which, located in  $A$ , moves with velocity  $\mathbf{v}_A$  and acceleration  $\mathbf{a}_A$  at the time  $t$ .

With regard to eq.(2.5), the velocity  $\mathbf{v}_A$  coincides with the term  $\mathbf{v}_t$  while the term  $\mathbf{v}_r$  is the relative velocity of  $B$  with respect to the mobile reference system. In fact, the rigid body is roto-translating, therefore, as the distance  $|\mathbf{BA}|$  is constant, the relative trajectory of  $B$ , with respect to an observer who is translating with the mobile reference system with origin in  $A$ , is a short arc of a circle whose centre is  $A$  and radius  $\rho$ . If the angular speed  $\omega$  is known, the eq.(2.5) allows one to evaluate the absolute velocity vector  $\mathbf{v}_B$  of any point of the rigid body.

The absolute acceleration of  $B$  can be obtained by differentiating the eq.(2.5):

$$\mathbf{a}_B = \rho \ddot{\vartheta} e^{i(\vartheta+\pi/2)} + \rho \vartheta^2 e^{i(\vartheta+\pi)} + \mathbf{a}_A = \rho \dot{\omega} e^{i(\vartheta+\pi/2)} + \rho \omega^2 e^{i(\vartheta+\pi)} + \mathbf{a}_A \quad (2.6)$$

where  $\dot{\omega}$  is the instantaneous angular acceleration with which the rigid body rotates, at the time  $t$  ( $\dot{\omega} = \ddot{\theta}$ ). By considering the translating coordinate system above mentioned, the eq.(2.6) can be rewritten as:

$$\mathbf{a}_B = \mathbf{a}_r + \mathbf{a}_t \tag{2.7}$$

$$\mathbf{a}_r = \mathbf{a}_{rt} + \mathbf{a}_{rn} = \rho \dot{\omega} e^{i(\theta + \pi/2)} + \rho \omega^2 e^{i(\theta + \pi)} \qquad \mathbf{a}_t = \mathbf{a}_A$$

where  $\mathbf{a}_r$  is the relative acceleration of B, with respect to the mobile reference system and  $\mathbf{a}_t$  is the *drag* acceleration. If the angular acceleration,  $\dot{\omega}$ , of the rigid body is known, the eqs.(2.6-2.7) allow one to evaluate the absolute acceleration vector  $\mathbf{a}_B$  of any point of the rigid body, see Figures 2.5 and 2.6.

The term  $\rho \omega^2 e^{i(\theta + \pi)}$  is the centripetal component of the relative acceleration  $\mathbf{a}_r$ : this vector is normal to the relative trajectory of B. Conversely, the term  $\rho \dot{\omega} e^{i(\theta + \pi/2)}$  is the tangential component of the relative acceleration  $\mathbf{a}_r$ : this vector is tangent to the relative trajectory of B.

If necessary, also the drag acceleration  $\mathbf{a}_t$  can be decomposed into the two vectors  $\mathbf{a}_{tn}$  and  $\mathbf{a}_{tt}$ , normal and tangent, respectively, to the absolute trajectory of the point A.

Therefore, the absolute velocity and acceleration vectors  $\mathbf{v}_B$  and  $\mathbf{a}_B$  of a generic point B of a rigid body can be evaluated by assuming that the following parameters are known, at the time  $t$ :

- the absolute velocity vector  $\mathbf{v}_A$  of a point A of the rigid body;
- the absolute acceleration vector  $\mathbf{a}_A$  of a point A of the rigid body;
- the absolute angular speed  $\omega$  of the rigid body;
- the absolute angular acceleration  $\dot{\omega}$  of the rigid body.

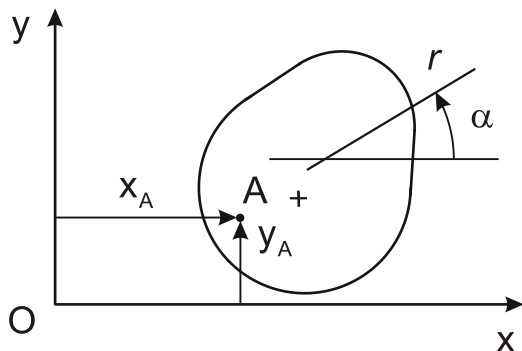


Figure 2.1 Kinematic analysis of a rigid body: basic parameters

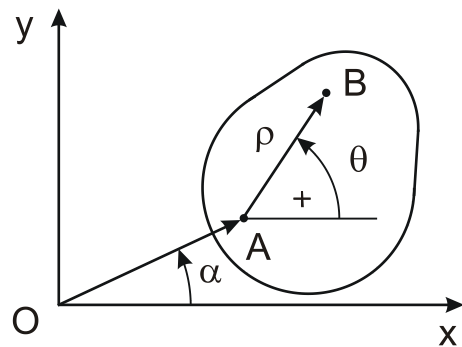


Figure 2.2 Kinematic analysis of a rigid body: position vectors

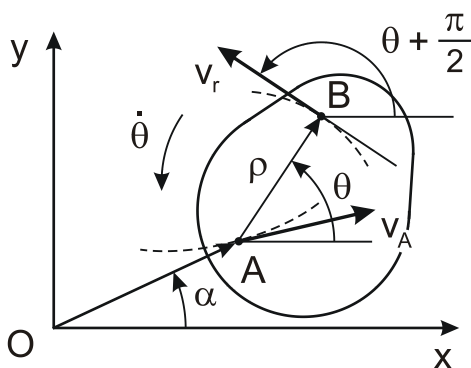


Figure 2.3 Kinematic analysis of a rigid body: relative velocity of the point B

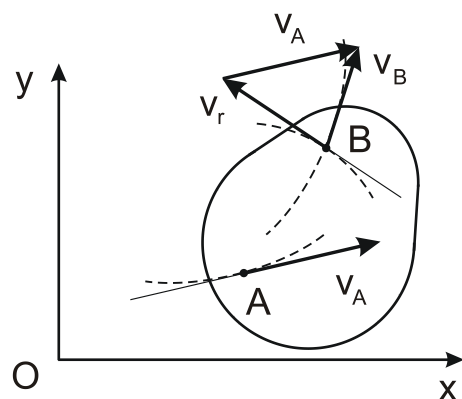


Figure 2.4 Kinematic analysis of a rigid body: absolute velocity of the point B

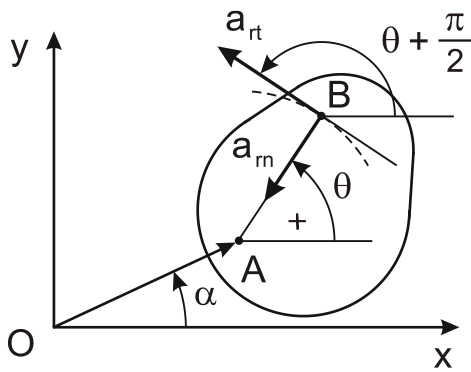


Figure 2.5 Kinematic analysis of a rigid body: relative acceleration of the point B

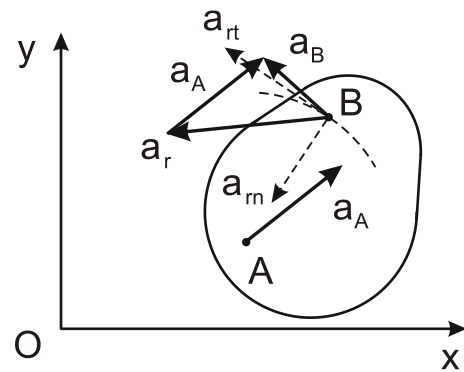


Figure 2.6 Kinematic analysis of a rigid body: absolute acceleration of the point B

## 2.1 Instant Centre of Rotation of a rigid body

The instant centre of rotation, also called *instantaneous centre* and *instant centre*, is the point in a body undergoing planar movement that has zero velocity at a given time  $t$ . At this instant the velocity vectors of the trajectories of other points in the body generate a circular field about this point which is identical to what is generated by a pure rotation.

Planar movement of a body is often described using a plane figure moving in a two dimensional plane. The instant centre is the point in the moving plane about which all other points are rotating at a specific instant of time. The continuous movement of a plane has an instant centre for every value of the time parameter. This generates a curve called the moving centrode. The points in the fixed plane corresponding to these instant centres form the fixed centrode.

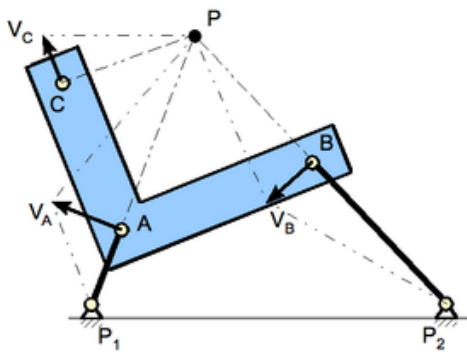


Figure 2.7 Instant centre of a moving plane

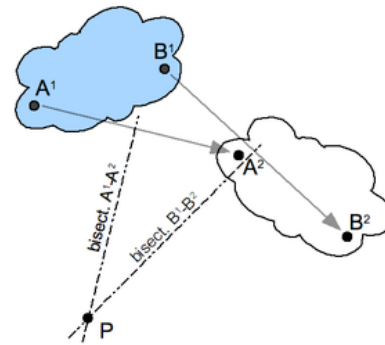


Figure 2.8: Pole of a planar displacement

### 2.1.1 Pole of a planar displacement

The instant centre can be considered the limiting case of the pole of a planar displacement. The planar displacement of a body from position 1 to position 2 is defined by the combination of a planar rotation and planar translation. For any planar displacement there is a point in the moving body that is in the same place before and after the displacement. This point is the *pole of the planar displacement*, and the displacement can be viewed as a rotation about this pole.

### 2.1.2 Construction for the pole of a planar displacement

First, select two points A and B in the moving body and locate the corresponding points in the two positions; see the illustration. Construct the perpendicular bisectors to the two segments  $A^1A^2$  and  $B^1B^2$  (Fig. 2.8) The intersection P of these two bisectors is the pole of the planar displacement. Notice that  $A^1$  and  $A^2$  lie on a circle about P. This is true for the corresponding positions of every point in the body.

If the two positions of a body are separated by an instant of time in a planar movement, then the pole of a displacement becomes the instant centre. In this case, the segments constructed between the instantaneous positions of the points A and B become the velocity vectors  $\mathbf{v}_A$  and  $\mathbf{v}_B$ . The lines perpendicular to these velocity vectors intersect in the instant centre.

### 2.1.3 Pure translation

If the displacement between two positions is a pure translation, then the perpendicular bisectors of the segments  $A^1B^1$  and  $A^2B^2$  form parallel lines. These lines are considered to intersect at a point on the line at infinity, thus the pole of this planar displacement is said to "lie at infinity" in the direction of the perpendicular bisectors. In the limit, pure translation becomes planar movement with point velocity vectors that are parallel. In this case, the instant centre is said to lie at infinity in the direction perpendicular to the velocity vectors.

### 2.1.4 Instant centre of a wheel rolling without slipping

Consider the planar movement of a circular wheel rolling without slipping on a linear road (Figure 2.9). The wheel rotates about its axis M, which translates in a direction parallel to the road. The point of contact P of the wheel with road does not slip, which means the point P has zero velocity (relative) with respect to the fixed plane (road). Thus, at the instant  $t$  the point P on the wheel comes in contact with the road. It becomes an instant centre of rotation of the wheel.

The set of points of the moving wheel that become instant centres is the circle itself, which defines the *moving centrode*. The points in the fixed plane that correspond to these instant centres is the line of the road, which defines the *fixed centrode*.

The velocity vector of a point A in the wheel is perpendicular to the segment AP and is proportional to the length of this segment. In particular, the velocities of points in the wheel are determined by the angular velocity of the wheel in rotation about P. The velocity vectors of a number of points are illustrated in Figure 2.9.

The further a point in the wheel is from the instant centre P, the proportionally larger its speed. Therefore, the point at the top of the wheel moves in the same direction as the centre M of the wheel, but twice as fast, since it is twice the distance away from P. All points that are a distance equal to the radius of the wheel 'r' from point P move at the same speed as the point M but in different directions. This is shown for a point on the wheel that has the same speed as M but moves in the direction tangent to the circle about P.

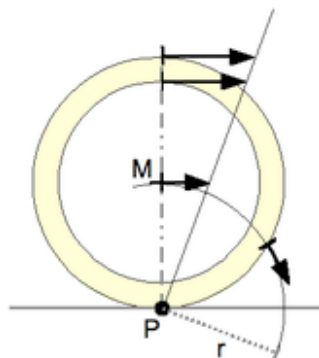


Figure 2.9 Rolling wheel

### 2.1.5 Additional information

Here below, some additional information about the Instant Centre of Rotation is reported. Let us consider the rigid body shown in Figure 2.10. The point A is moving along the absolute trajectory  $s_A$  with the absolute velocity  $\mathbf{v}_A$ , at time  $t$ . This vector is tangent to the trajectory  $s_A$  (Figure 2.10). Then, let us consider a generic point B of the rigid body that is moving along the absolute trajectory  $s_B$ , at time  $t$ . The absolute velocity of B is unknown but its direction coincides with the straight line tangent in B to  $s_B$  (Figure 2.11).

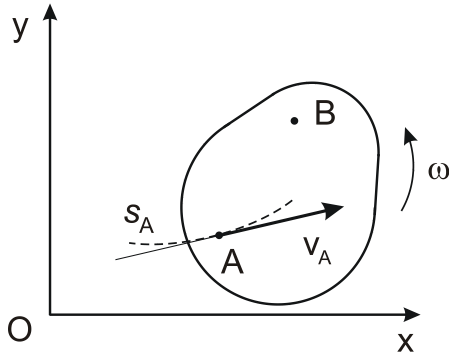


Figure 2.10 Motion of a rigid body in a 2-D space

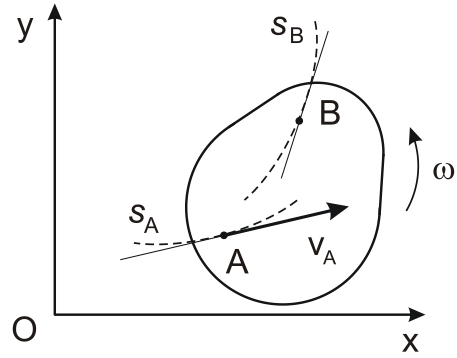


Figure 2.11 Motion of a rigid body in a 2-D space

Let us denote  $P_0$  the Instant Centre of Rotation (ICR) of the rigid body (in the absolute motion), at time  $t$ . Being  $P_0$  a ICR, its absolute velocity is null:  $\mathbf{v}_{P_0} = 0$ . By considering a translational coordinate system with origin in  $P_0$  we have:

$$\mathbf{v}_A = \mathbf{v}_{P_0} + \boldsymbol{\omega} \wedge (\mathbf{A} - \mathbf{P}_0) \quad \text{and} \quad \mathbf{v}_B = \mathbf{v}_{P_0} + \boldsymbol{\omega} \wedge (\mathbf{B} - \mathbf{P}_0) \quad (2.8)$$

That is:

$$\mathbf{v}_A = \boldsymbol{\omega} \wedge (\mathbf{A} - \mathbf{P}_0) \quad \text{and} \quad \mathbf{v}_B = \boldsymbol{\omega} \wedge (\mathbf{B} - \mathbf{P}_0) \quad (2.9)$$

In accordance with the cross-product rules, the vector  $\mathbf{v}_A$  is orthogonal to the vector  $(\mathbf{A} - \mathbf{P}_0)$ , while the vector  $\mathbf{v}_B$  is orthogonal to the vector  $(\mathbf{B} - \mathbf{P}_0)$ . Therefore, the ICR  $P_0$  is given by the intersection between the straight lines passing through A and B, which are normal to the trajectories  $s_A$  and  $s_B$  (Figure 2.12).

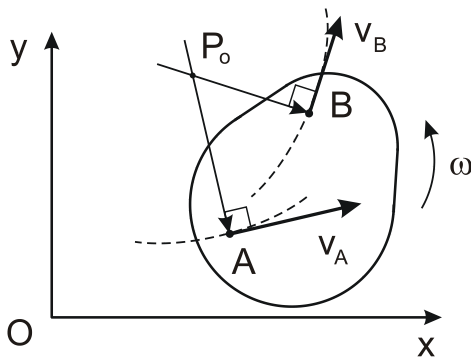


Figure 2.12 Instant Centre of Rotation (ICR),  $P_0$

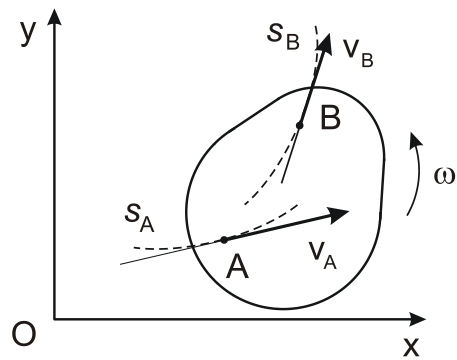


Figure 2.13 Absolute velocity of the point B

Therefore, when the absolute velocity vector  $\mathbf{v}_B$  of a generic point, B, of a rigid body must be evaluated, the following two basic case studies may occur:

- 1) the known data of a kinematical problem are:
  - the coordinates of the ICR position,  $P_0$ ;
  - the instantaneous value of the angular speed,  $\boldsymbol{\omega}$ , of the body; (in this case study the absolute trajectory of the point B, and then the direction of the velocity  $\mathbf{v}_B$ , are unknown);
- 2) the known data of a kinematical problem are:
  - the absolute velocity vector  $\mathbf{v}_A$  of a point, A, of the body;
  - the absolute trajectory of the point B, the absolute velocity of which must be evaluated.

In the case (1) the velocity vector  $\mathbf{v}_B$  is given by:  $\mathbf{v}_B = \boldsymbol{\omega} \wedge (\mathbf{B} - \mathbf{P}_0)$ .

In the case (2) the preliminary evaluation of the position of the ICR  $P_0$  must be carried out. This problem can be solved by evaluating the coordinates of the intersection between the straight line,  $r_1$ , that is orthogonal, in A, to the velocity vector  $\mathbf{v}_A$  and the straight line,  $r_2$ , passing from the point B, that is orthogonal to the straight line tangent, in B, to the absolute trajectory of B (Figure 2.12). The intersection point between the two straight lines  $r_1$  and  $r_2$  is the ICR  $P_0$ . After having determined the position of  $P_0$  it is possible to evaluate the angular speed  $\omega$  of the rigid body from the following expression:  $\mathbf{v}_A = \omega \wedge (\mathbf{A} - \mathbf{P}_0)$ .

Then, the absolute velocity vector  $\mathbf{v}_B$  can be obtained by:  $\mathbf{v}_B = \omega \wedge (\mathbf{B} - \mathbf{P}_0)$ , (Figure 2.13).

As said above, the instant centre of rotation,  $P_0$ , of a rigid body subjected to a planar motion is a point of the plane that, for an infinitesimal time interval about the current time, can be considered rigidly connected to the body itself. The point  $P_0$  can be internal or external to the rigid body. The instant centre of rotation can be defined for the *absolute motion* of the body and for the *relative motion* as well. When the ICR is defined for the *absolute motion* of body, its instantaneous absolute velocity,  $\mathbf{v}_{P_0}$ , is null. Conversely, when the ICR is defined for the *relative motion* of body, its instantaneous relative velocity,  $\mathbf{v}_{rP_0}$ , is null.

### 2.1.5.1 Relative centre of rotation for two contacting planar bodies

If two planar rigid bodies are in contact, and each body has its own distinct centre of rotation, then the relative centre of rotation between the bodies has to lie somewhere on the line connecting the two centres. As a result since pure rolling can only exist when the centre of rotation is at the point of contact (as seen above with the wheel on the road), it is only when the point of contact goes through the line connecting the two rotation centres that pure rolling can be achieved. This is known in *Involute gear* design as the pitch point, where there is no relative sliding between the gears. In fact, the gearing ratio between the two rotating parts is found by the ratio of the two distances to the relative centre. In the example in Figure 2.14 the gearing ratio is:  $\gamma = AD/BD$ .

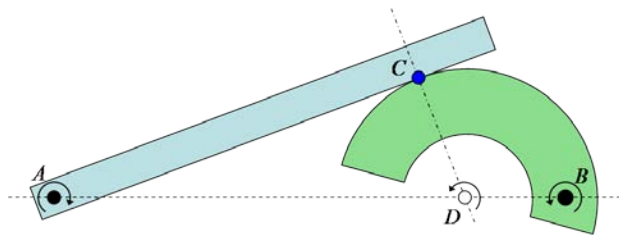


Figure 2.14 Example of relative centre of rotation. Two bodies in contact at C, one rotating about A and the other about B must have a relative centre of rotation somewhere along the line AB. Since the parts cannot interpenetrate the relative rotation centre must also be along the normal direction to the contact and through C. The only possible solution is if the relative centre is at D.

Consider the Instantaneous Centre of Rotation for the relative motion of the circular-shaped body hinged in B, with respect to the beam hinged in A. The relative velocity of the contact point C, with respect to the beam, is tangent to the beam in C. That is, it is parallel to the straight line passing through A and C (the contact point C moves along the beam profile. For an observer fixed to the beam, with origin in A, point C moves along a rectilinear straight line passing through points A and C). The relative trajectory of point B, with respect to the beam, must be a circle of centre A and radius AB, as the distance AB is constant (for the above mentioned observer – who is rotating with the beam – the motion of point B, with respect to the observer, is a circle of radius AB and centre A). Therefore, the relative velocity of B, with respect to the beam, is orthogonal to the straight line passing through AB. Thus, the relative ICR, D, of the circular-shaped body is the intersection of the straight lines AB and CD, which are orthogonal to the above mentioned relative velocity vectors of points B and C (with respect to the beam hinged in A).

The ICR, D, has a null relative velocity, in the relative motion of the circular-shaped body hinged in B, with respect to the beam hinged in A. Then, if we consider a mobile reference system, we can write:

$$\mathbf{v}_D = \mathbf{v}_r + \mathbf{v}_t \quad \mathbf{v}_D = \mathbf{v}_{rD} + \mathbf{v}_{dD} \quad (2.10)$$

That is, the absolute velocity of point D is the sum of a relative velocity vector and a drag velocity vector. Being the relative velocity  $\mathbf{v}_{rD}$  null, we obtain:  $\mathbf{v}_D = \mathbf{v}_{dD}$ .

This means that the velocity of point D, assumed fixed to the beam (a rotating reference frame with origin in A, fixed to the beam has been considered) coincides with the velocity of the same point assumed to be fixed to the circular-shaped body (a rotating reference frame with origin in B, fixed to the circular-shaped body has been considered).

Let us denote as  $\omega_1$  and  $\omega_2$  the angular velocity of the circular-shaped body and beam, respectively (see Figure 2.14). The angular velocity  $\omega_2$  is clockwise (negative), while the angular velocity  $\omega_1$  is counter-clockwise (positive). Then, the absolute velocity of point D, assumed as a part of the circular-shaped body, can be written as:

$$\mathbf{v}_D = \omega_1 \wedge (\overline{D-B}) \quad (2.11)$$

The absolute velocity of point D, assumed to be part of the beam, can be written as:

$$\mathbf{v}_D = \omega_2 \wedge (\overline{D-A}) \quad (2.12)$$

Therefore, we can write:

$$\omega_2 = \omega_1 \frac{|\overline{D-B}|}{|\overline{D-A}|} \quad (2.13)$$

In the end, the relative velocity of point C, with respect to the beam, is given by:

$$\mathbf{v}_{rC} = \omega_r \wedge (\overline{C-D}) \quad (2.14)$$

where the relative angular velocity  $\omega_r$  can be obtained assuming to assign to the system a motion opposite to that of the beam. This way, the beam remains fixed while the circular-shaped body describes a relative motion, with respect to the beam, with the following relative angular velocity:

$$\omega_1 = \omega_r + \omega_2 \quad \text{that is:} \quad \omega_r = \omega_1 - \omega_2 \quad (2.15)$$

Note that the angular velocities  $\omega_1$  and  $\omega_2$  are opposite and that the magnitude of  $\omega_1$  is higher than that of  $\omega_2$  (DB is smaller than DA). Therefore, the relative angular velocity  $\omega_r$  is positive (counter-clockwise), that is vector  $\mathbf{v}_{rC}$  is directed leftwards (from point C to point A). This result is also intuitive.

Figure 2.15 shows a further example of an instant centre of rotation in the relative motion. The disk is rolling on the inclined plane of a moving wedge, at an angle  $\alpha$  to the horizontal direction. The wedge is moving with an absolute velocity  $\mathbf{v}$ . The angular speed of the disk is  $\omega$ . We assume that no slipping occurs between the disk and the supporting plane. The contact point H, between the disk and the plane, has a null relative velocity.

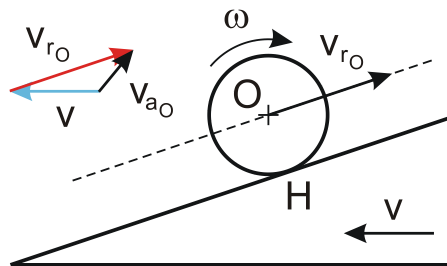


Figure 2.15 Example of relative centre of rotation.

Consider a translational reference system connected to the wedge, which is a reference system that translate with the wedge velocity  $\mathbf{v}$ . The relative trajectory of point O, with respect to the mobile reference system, is a straight line parallel to the inclined surface of the wedge. The relative velocity of point O is given by:

$$\mathbf{v}_{rO} = \boldsymbol{\omega} \wedge (\mathbf{O} - \mathbf{H}) \quad (2.16)$$

Then, the absolute velocity is given by:

$$\mathbf{v}_{aO} = \mathbf{v}_{rO} + \mathbf{v} \quad (2.17)$$

### 3. KINEMATICS OF A ROTATING DISK

Let us consider a purely rotating disk that is moving on a rectilinear surface. Let us denote H the contact point between disk and plane, and by  $\boldsymbol{\omega}$  the disk angular velocity (Figure 3.1). As said above, it is assumed that no slippage occurs between the disk and the supporting plane. As a consequence of this, the relative velocity of the point H, with respect to the guide, is null. The absolute velocity of the point H<sub>1</sub>, assumed belonging to the disk, is equal to the absolute velocity of the point H<sub>2</sub>, assumed belonging to the guide. If the guide is fixed, the absolute velocity of the point H is null. Therefore, the point H is the Instant Centre of Rotation (ICR) of the disk, at time  $t$ . Then, the velocity of the disk centre, O, is given by:

$$\mathbf{v}_O = \boldsymbol{\omega} \wedge (\mathbf{O} - \mathbf{H}) \quad (3.1)$$

The absolute trajectory of the point O is a straight line, parallel to the guide (Figure 3.1). The direction and versus of the velocity  $\mathbf{v}_O$  is given by the versor  $\mathbf{t}$  (Figure 3.2), the direction of which is constant.

Let us evaluate the velocity and acceleration vectors of a generic point A of the disc. By considering a translating reference system,  $x$ - $y$ , with origin in O we can write:  $\mathbf{v}_t = \mathbf{v}_O$ .

In fact, the *drag* velocity  $\mathbf{v}_t$  coincides with  $\mathbf{v}_O$ . Besides, the disk is a roto-translating rigid body. Therefore, its motion relative to an observer who is fixed with the translating coordinate system  $x$ - $y$  is only a pure rotation with angular speed  $\boldsymbol{\omega}$ . The trajectory of A, relative to the mobile reference system, is a short circular arc whose centre and radius are O and  $(\mathbf{A} - \mathbf{O})$ , respectively. Therefore, the absolute velocity of A is given by:

$$\mathbf{v}_A = \mathbf{v}_t + \mathbf{v}_r = \mathbf{v}_O + \boldsymbol{\omega} \wedge (\mathbf{A} - \mathbf{O}) \quad (3.2)$$

As H is the disc ICR, the velocity vector  $\mathbf{v}_A$  (Figure 3.4) is also given by:

$$\mathbf{v}_A = \boldsymbol{\omega} \wedge (\mathbf{A} - \mathbf{H}) \quad (3.3)$$

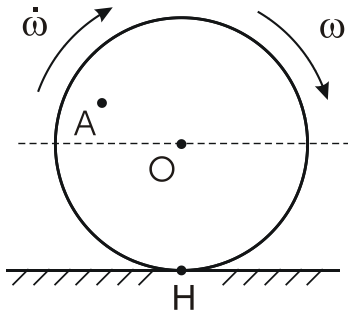


Figure 3.1 Rotating disk

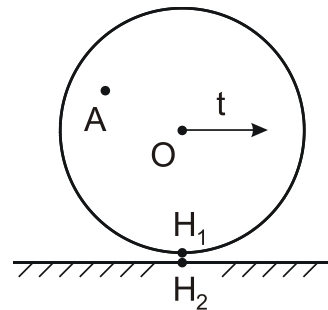


Figure 3.2 Rotating disk: contact point

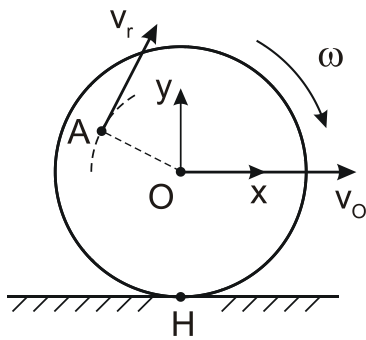


Figure 3.3 Rotating disk: relative velocity of point A with respect to the translating reference system  $x$ - $y$

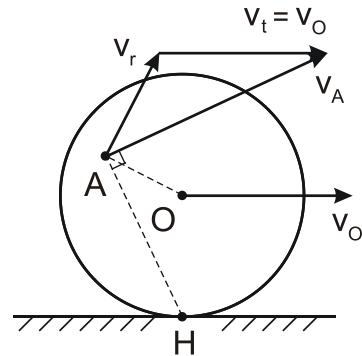


Figure 3.4 Velocity of point A, according to Rivals' theorem and ICR method

The relative acceleration  $\mathbf{a}_r$  can be decomposed into two vectors  $\mathbf{a}_{tn}$  and  $\mathbf{a}_{tt}$  that are normal and tangent, respectively, to the relative trajectory of the point A, with respect to the mobile reference system. In accordance with Rivals' theorem, the absolute acceleration of A can be written as:

$$\mathbf{a}_A = \mathbf{a}_t + \mathbf{a}_r = \mathbf{a}_t + (\mathbf{a}_{tn} + \mathbf{a}_{tt}) = \mathbf{a}_O + \boldsymbol{\omega} \wedge [\boldsymbol{\omega} \wedge (\mathbf{A} - \mathbf{O})] + \dot{\boldsymbol{\omega}} \wedge (\mathbf{A} - \mathbf{O}) \quad (3.4)$$

The components of the acceleration vector of point A are shown in Figure 3.5. The absolute acceleration vector of point A is shown in Figure 3.6.

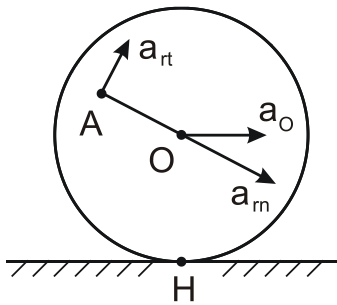


Figure 3.5 Components of the acceleration vector of point A

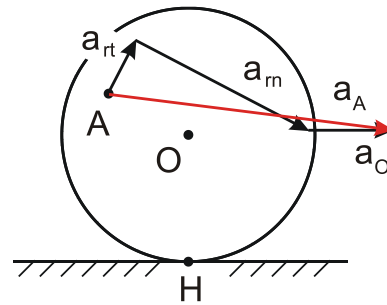


Figure 3.6 Absolute acceleration of point A

Beware of the fact that the ICR method cannot be used to evaluate acceleration vectors. Therefore, the acceleration of A must be evaluated using a mobile reference system (or alternative methods).

#### 4. INSTANT CENTRE OF ROTATION: AN EXAMPLE

Let us consider the bar AB shown in Figure 4.1. The end A is in contact with a horizontal plane while the end B is in contact with an inclined plane. At the given time the absolute velocity of point A is  $\mathbf{v}_A$ . We want to evaluate the absolute velocity of point B with a method based on the Instant Centre of Rotation (ICR).

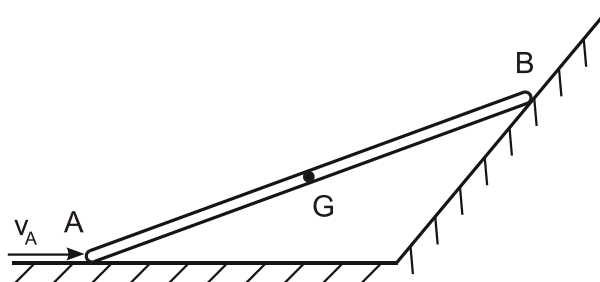


Figure 4.1 System configuration

The magnitude of the absolute velocity of point B is unknown but its direction is known. In fact, the absolute trajectory of point B is a straight line parallel to the inclined plane, passing through point B. Therefore, the Instant Centre of Rotation, P, of the bar AB can be obtained as the intersection between the vertical straight line passing through point A, orthogonal to the horizontal velocity vector  $\mathbf{v}_A$ , and the straight line passing through point B, orthogonal to the inclined plane, that is orthogonal to the velocity vector  $\mathbf{v}_B$ , whose magnitude is unknown (Figure 4.2). At any given time, the absolute velocity of point P is null (it is the ICR of the rigid body represented by the bar!).

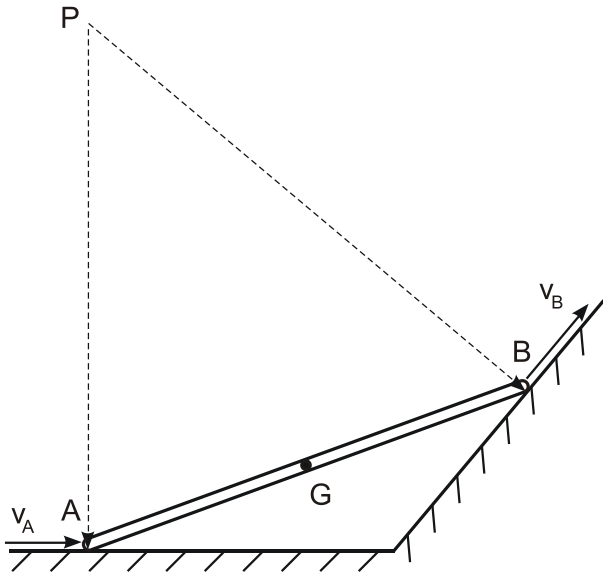


Figure 4.2 ICR of the bar AB and absolute velocity vector of point B

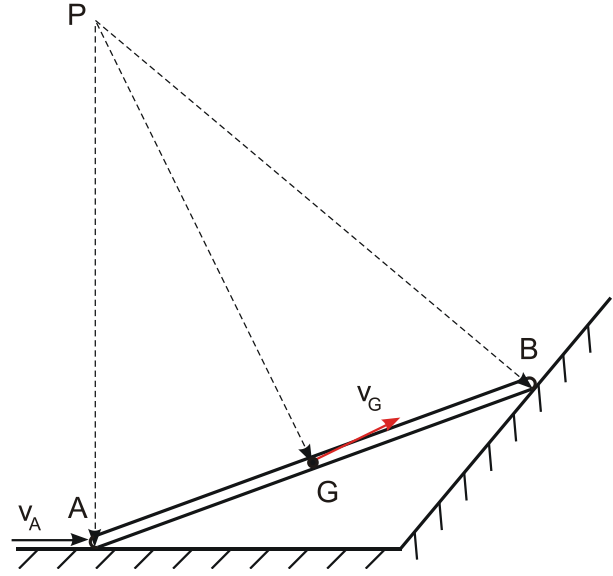


Figure 4.3 ICR of the bar AB and absolute velocity vector of point G

The bar AB is subjected to a roto-translational motion. The unknown absolute angular velocity of the bar can be denoted  $\boldsymbol{\omega}$ . Based on the properties of the ICR it is possible to write:

$$\mathbf{v}_A = \boldsymbol{\omega} \wedge \overline{(A - P)} \quad (4.1)$$

This way, it is possible to evaluate the magnitude of the angular velocity of the bar:

$$|\boldsymbol{\omega}| = \frac{|\mathbf{v}_A|}{|\overline{(A - P)}|} \quad (4.2)$$

The direction of the angular velocity can be obtained by means of the cross-product of eq.(4.1). Then, the absolute velocity of point B is given by:

$$\mathbf{v}_B = \boldsymbol{\omega} \wedge \overline{(B - P)} \quad (4.3)$$

The absolute velocity  $\mathbf{v}_G$  of the barycentre G of the bar AB can be expressed as:

$$\mathbf{v}_G = \boldsymbol{\omega} \wedge \overline{(G - P)} \quad (4.4)$$

The direction of the velocity vector  $\mathbf{v}_G$  is orthogonal to the vector  $\overline{(G - P)}$ , as shown in Figure 4.3.

It is necessary to keep in mind that the method used to evaluate the velocity vectors  $\mathbf{v}_B$  and  $\mathbf{v}_G$  cannot be used to evaluate the corresponding acceleration vectors  $\mathbf{a}_B$  and  $\mathbf{a}_G$ . In fact, the absolute acceleration of

point P is not necessarily null. Figure 4.4 shows the position of point P at different time instants. You can see that point P is not fixed as it is not the absolute centre of the acceleration. Therefore, the absolute acceleration of points B and G must be evaluated with different techniques.

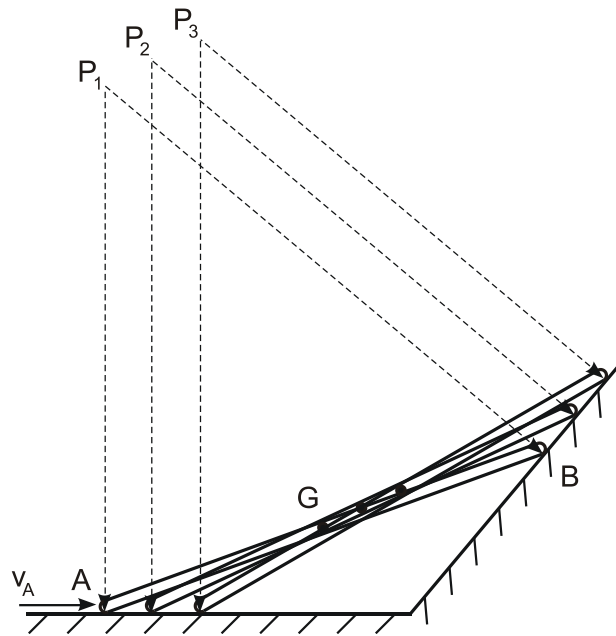


Figure 4.4 Evolution over time of the position of the ICR P of the bar AB